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THE INFLUENCE OF A STOCHASTIC ENVIRONMENT ON THE FIRM'S OPTIMAL DYNAMIC INVESTMENT POLICY

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In this paper we study the impact of an uncertain environment on the optimal dynamic investment policy of a value maximizing firm. We analyse a model of Bensoussan & Lesourne (1980) and extend their conclusions in the mathematical solution as well as in the economic interpretation. The relation between the shareholder's attitude towards risk and uncertainty about the firm's outcome turns out to play an important role for the firm's optimal dynamic policy. Moreover, we extend the model by incorporating the price of risk (wellknown from the static Capital Asset Pricing Model) into the shareholder's time preference rate.

1. INTRODUCTION

Empirical studies have shown that the development of the firm over time can be divided into different stages, characterised by growth, stationarity and contraction. In order to understand, evaluate and control these stages, economists have used dynamic mathematical techniques like optimal control theory, calculus of variations and dynamic programming to develop and analyse dynamic models of the firm. In this way, the dynamic theory of the firm has been a fruitful area for interesting scientific contributions.

Jorgenson (1963) was the first to apply the maximum principle in order to analyse the dynamics of the firm. However, his optimal solution showed an unrealistic immediate adjustment of the capital good stock to the level of maximum revenue. Later scientists introduced two ways to avoid this "jump" at the start of the planning period, in particular by the introduction of financing limits (e.g. Lesourne (1973), Ludwig (1978) and Van Loon (1983)) and by incorporating adjustment costs (e.g. Gould (1968), Nickell (1978) and Treadway (1969)). Van Schijndel (1986, 1987) extended the financial models by introducing taxation and Kort (1988) incorporated both financing limits and adjustment costs in one model. The book of Feichtinger & Hartl (1986) contains a good survey of these kinds of models.

The above models all have in common that they assume a certain future. The purpose of this paper is to extend these studies by adding another dimension : uncertainty. Our article leans heavily on the pathbreaking work of Bensoussan & Lesourne (1980, 1981). Using the technique of dynamic programming they prove, that - depending on the firm's capital good stock, its amount of cash, the uncertainty of the firm's investment and the time preference rate of the shareholder - the firm makes a choice between three destinations of its revenue : increase the amount of cash, invest the money or pay out dividend. In this paper we extend the analysis of Bensoussan & Lesourne by introducing criteria for the different optimal solutions, improving some of the solutions and adding economic interpretations.

At last we argue that the model is incorrect from an economic point of view : from the static theory of the Capital Asset Pricing Model we know that the shareholder wants a compensation in the firm's rate of return for the risk he takes. Therefore, the "price of risk" must be incorporated into the shareholder's time preference rate. We also analyse the implications of this extension for the optimal solution.

2. THE MODEL

In this section we present the stochastic dynamic model of the firm of Bensoussan & Lesourne (1980).

The stochastic aspect of the model is incorporated in the earnings function, which is expressed as :

$$R(K(T)) = S(K(T))\ (1 + \sigma V(T)) \tag{1}$$

in which :

- T = time
- $K(T)$ = capital good stock
- $R(K(T))$ = earnings function
- $S(K(T))$ = deterministic earnings function, $\frac{dS}{dK} > 0$, $\frac{d^2S}{dK^2} < 0$, $S(0) = 0$, $\left.\frac{dS}{dK}\right|_{K=0} > i$
- $V(T)$ = Gaussian stochastic variable, $E(V(T)) = 0$, $VAR(V(T)) = 1$, $E(V(T),V(\hat{T})) = 0$ if $T \neq \hat{T}$
- σ = a constant

From (1) we derive that the expected earnings are equal to $S(K)$, the variance is $\sigma^2(S(K))^2$ and the disturbances are independently distributed over time. $V(T)$ is the "white noise".

In order to apply the technique of dynamic programming, we rewrite (1) into a stochastic differential :

$$R(K(T))dT = S(K(T))\ (dT + \sigma dB(T)) \tag{2}$$

in which :

$B(T)$ = a standard Wiener process with $B(0) = 0$ and independent increments $dB(T)$ normally distributed with zero mean and variance dT.

Given the uncertain situation we need a "buffer" in order to compensate the possible "disappointing earnings" and therefore, we incorporate cash in the model. This is a main difference compared to the deterministic dynamic models of the firm. Now the balance sheet becomes :

$$X(T) = K(T) + M(T) \tag{3}$$

in which :

$M(T)$ = cash

$X(T)$ = equity

Further, we make the assumption that the firm is bankrupt as soon as $M(T)$ becomes negative. As for the objective, we suppose that the firm behaves as if it maximizes the shareholder's value of the firm. This value consists of the mathematical expectation of the discounted distributed dividends over the planning period, so :

$$\text{maximize} : E\left(\int_0^Z D(T)\, e^{-iT}\, dT \right) \tag{4}$$

in which :

$D(T)$ = dividend

Z = planning horizon

In this model Z is equal to the bankruptcy time, be the first instant reached for which $M(T) < 0$.

i = time preference rate of the shareholder

Earnings are used to increase the net assets or to pay out dividend :

$$\dot{X}(T) = R(K(T)) - D(T) \tag{5}$$

Cash has to be positive or zero, since otherwise the firm would go bankrupt:

$$M(T) \geq 0 \quad (6)$$

As far as its dividend policy is concerned, we assume that the firm is allowed to pay no dividend, so :

$$D(T) \geq 0 \quad (7)$$

We further assume that there are no depreciations and that past investment cannot be sold (perfect irreversibility of investment) :

$$\dot{K}(T) \geq 0 \quad (8)$$

In order to get the payoff (4) to be well defined , the controls have to be bounded by deterministic constants. To achieve this, we assume that at any time, dividend is not greater than the difference between the expected earnings and investment :

$$D(T) \leq S(K(T)) - \dot{K}(T) \quad (9)$$

Note, that (2), (3), (5) and (9) imply that the mathematical expectation of $\dot{M}$ has to be greater or equal to zero. In the deterministic case (i.e. σ = 0) M would be zero implying (9) to be an equality, which means that at any time income must be equal to expenses.

We finally assume that the initial values of K and M are positive :

$$K(0) = K_0 > 0 \quad (10)$$

$$M(0) = M_0 > 0 \quad (11)$$

3. OPTIMAL POLICIES

First , we introduce :

$$U(T) = \max E \left(\int_T^Z D(t)\, e^{-it} dt \right) \quad (12)$$

U(T) is equal to the maximization of the mathematical expectation of the discounted dividend-stream of a firm which is characterised at a point of time T by M(T) and K(T) : U(T) = U(M(T),K(T)).

Using dynamic programming we can derive that the function U(T) is given by a partial differential equation , the Hamilton-Jacobi-Bellman equation :

$$iU = \max_{\substack{\dot{K}, D \geq 0 \\ \dot{K}+D \leq S(K)}} \{ D + \dot{K} \frac{\partial U}{\partial K} + (S(K) - \dot{K} - D) \frac{\partial U}{\partial M} \} + \frac{\sigma^2}{2} (S(K))^2 \frac{\partial^2 U}{\partial M^2} \qquad (13)$$

For the derivation of (13) we refer to Bensoussan & Lesourne (1980, pp 244 - 245), while Bensoussan & Lesourne (1981, section 3) contains a concise survey of the mathematical technique of stochastic dynamic systems.

This equation may be rewritten :

$$iU = S(K) \max \{ 1 , \frac{\partial U}{\partial K} , \frac{\partial U}{\partial M} \} + \frac{\sigma^2}{2} (S(K))^2 \frac{\partial^2 U}{\partial M^2} \qquad (14)$$

to which we adjoin the boundary equation :

$$U(0,K) = 0 \qquad (15)$$

To obtain the optimal solution , we have to solve the equations (14) and (15). Depending on the relative size of 1 , $\frac{\partial U}{\partial K}$ and $\frac{\partial U}{\partial M}$, one of the following three policies is optimal:

<u>Cash Policy</u> : $\dot{M} = S(K) (1 + \sigma V(T))$, $D = 0$, $\dot{K} = 0$

optimal if : $\frac{\partial U}{\partial M} \geq \max \{ 1, \frac{\partial U}{\partial K} \}$

(14) becomes :

$$iU = S(K) \frac{\partial U}{\partial M} + \frac{\sigma^2}{2} (S(K))^2 \frac{\partial^2 U}{\partial M^2} \qquad (16)$$

After solving the differential equation (16) we get :

$$U = k_1(K) \exp(r_1 \frac{M}{S(K)}) + k_2(K) \exp(r_2 \frac{M}{S(K)}) \qquad (17)$$

in which :

$k_1(K)$ and $k_2(K)$ are unknown functions

$$r_1 = [-1 + (1 + 2\sigma^2 i)^{1/2}] / \sigma^2 \qquad (18)$$

$$r_2 = [\ -1 - (\ 1 + 2\sigma^2 i\ )^{1/2}\ ] / \sigma^2 \tag{19}$$

Investment Policy : $\dot{M} = \sigma S(K)V(T)$, $D = 0$, $\dot{K} = S(K)$

Optimal if $\frac{\partial U}{\partial K} \geq \max\ \{\ 1,\ \frac{\partial U}{\partial M}\ \}$

(14) becomes :

$$iU = S(K)\ \frac{\partial U}{\partial K} + \frac{\sigma^2}{2}\ (S(K))^2\ \frac{\partial^2 U}{\partial M^2} \tag{20}$$

Equation (20) is partial differential equation , which we cannot solve.

Dividend Policy : $\dot{M} = \sigma S(K)V(T)$, $D = S(K)$, $\dot{K} = 0$

Optimal if $1 \geq \max\ \{\ \frac{\partial U}{\partial K},\ \frac{\partial U}{\partial M}\ \}$

(14) now becomes :

$$iU = S(K) + \frac{\sigma^2}{2}\ (S(K))^2\ \frac{\partial^2 U}{\partial M^2} \tag{21}$$

The solution of the differential equation is equal to :

$$U = \frac{S(K)}{i} + c_1(K)\ \exp(\ \frac{M\sqrt{(2i)}}{\sigma S(K)}\ ) + c_2(K)\ \exp(\ -\ \frac{M\sqrt{(2i)}}{\sigma S(K)}\ ) \tag{22}$$

in which c_1 and c_2 are unknown functions of K.

4. OPTIMAL SOLUTION

From (4) we can conclude that if the time preference rate i is great, the shareholder does not assign a high value to dividends in the far future. Therefore he wants to obtain dividends as soon as possible, even though the firm then has a bigger chance to go bankrupt. So, he does not like to invest or raise the amount of cash first in order to receive more dividends in the future, because there is risk of the firm going bankrupt before he has collected any dividends yet. In this sense, the shareholder is a "risk-averter". So, we can use $1/i$ as a measure of the degree to which the shareholder likes risk .

Depending on the relation between this measure of the shareholder's attitude towards risk ($1/i$) and the uncertainty of the firm's investment ($\sigma^2/2$), or equivalent, the relation between $1/i$ and $\sigma/\surd(2i)$, we can distinguish five optimal solutions, which will each be represented in the plane (M,K) (see figure (4.1)). The first four solutions are proved in Appendix 1 and for the last solution (the certainty case) we refer to Kort (1986).

At a certain point of time the firm always has a position in the plane (M,K) and depending on this position, and the corresponding optimal solution, it is optimal to carry out one of the three policies we derived in the previous section. In this way the evolution of the firm through time corresponds to a "random walk" in R^2. The firm switches to a different policy every time a border between two regimes is crossed.

Consider figure 4.1. In Panel A, the shareholder does not want to increase the amount of cash, even when cash is almost zero, because the firm's investment is too risky (i.e. σ is too high). Therefore he wants to obtain dividend as soon as possible because of the risk that the firm goes bankrupt. Above the horizontal line $K = K^*$, the firm never invests, because then the marginal return is lower than the shareholder's time preference rate, i.e. $\frac{dS}{dK} < i$ due to the concavity of S(K). This statement also holds for the other solutions.

In Panel B, the relation between the shareholder's likeliness of risk and the uncertainty of the investment outcome is such that the shareholder wants to take the risk of increasing the amount of cash first before receiving any dividends. The solution in Panel C is the limiting case of the second and fourth solution. Here the boundaries of "cash-dividend" and "dividend-investment" intersect in (0,0). In this way the area of the "dividend-region" decreases. The economic reason for this is that the shareholder now wants the firm to grow first before it pays out dividend. This statement holds in a stronger way in the solution of Panel D, which tends to a situation without risk. In Appendix 1 we prove that the boundary between the "cash-region" and the "investment-region" starts in the origin. This result disproves figure 8.4b in Bensoussan & Lesourne (1980).

In Panel E, the uncertainty of the investment is zero. Therefore, the use of cash as a buffer against bankruptcy is not necessary anymore. The firm invests when the marginal rate of return is higher than "i" and in the other region it pays out dividend.

For the "cash-region" we can conclude from the above that it does not exist

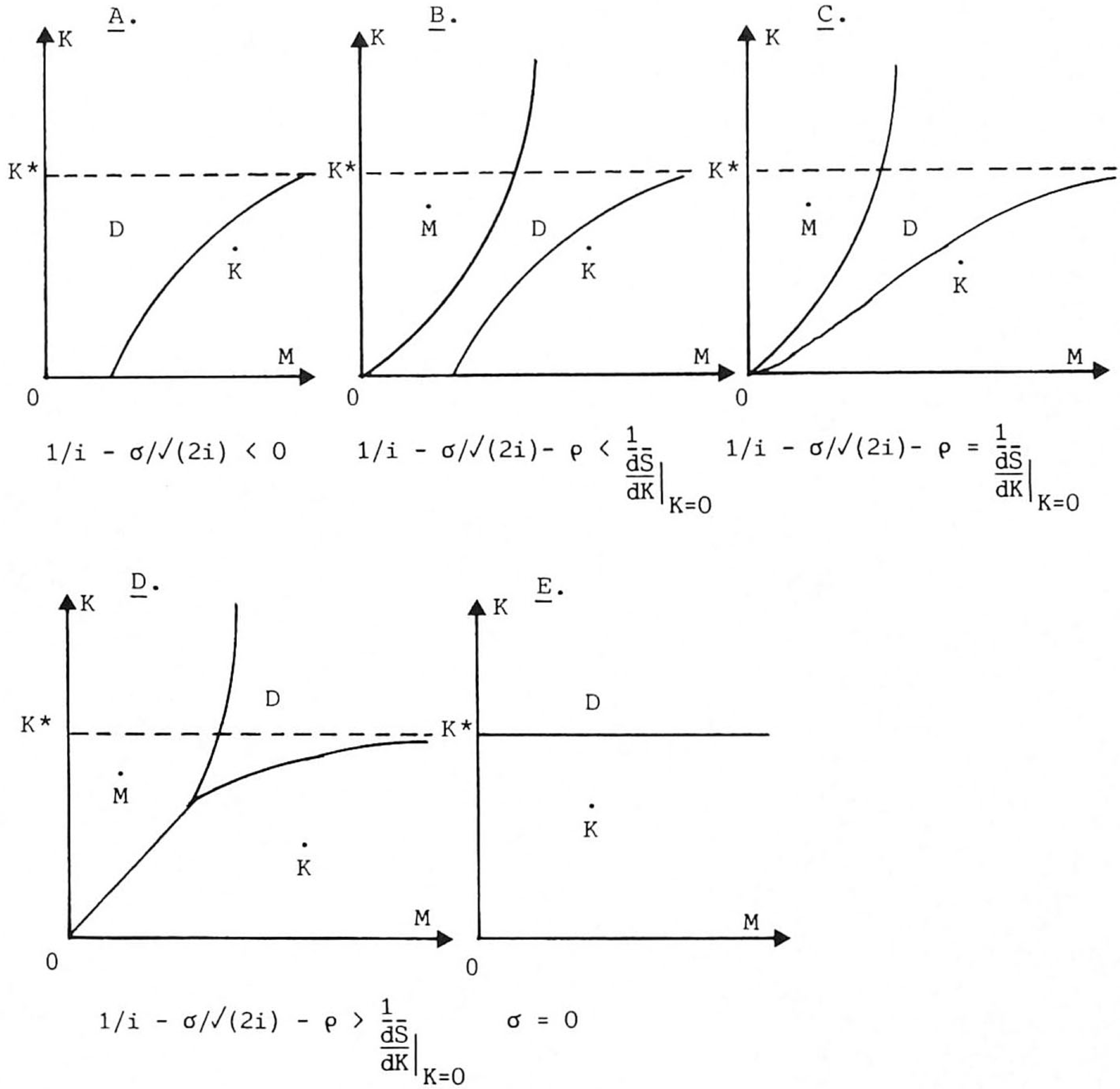

FIGURE 4.1

The five different solutions depending on the different relationships between $1/i$ and $\sigma/\sqrt{(2i)}$.

in which

D = dividend policy

$\dot{K}$ = investment policy

$\dot{M}$ = cash policy

$\left.\frac{dS}{dK}\right|_{K=K^*} = i$

ρ = a constant which satisfies the next relation (see Appendix 1) :

$$\exp((r_1 - r_2)\rho) = [1 - r_2(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}})]/[1 - r_1(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}})] \quad (23)$$

In Appendix 1 we further prove that the boundary between the "dividend-region" and the "cash-region" has the following mathematical expression :

$$M = \rho S(K) \tag{24}$$

Due to the fact that $S(0) = 0$, this boundary starts in $(0,0)$, which does not agree with the figures 8.2 and 8.4a in Bensoussan & Lesourne (1980).

when the investment is very risky, then it appears when risk decreases and it tends to disappear again when σ is very small, since in the certain case ($\sigma = 0$) one has only the dividend policy or the investment policy. Keeping in mind that the level of ρ determines the area of the "cash-region" (see equation (24)), this result coincides with figure 4.2, which is proved in Appendix 2.

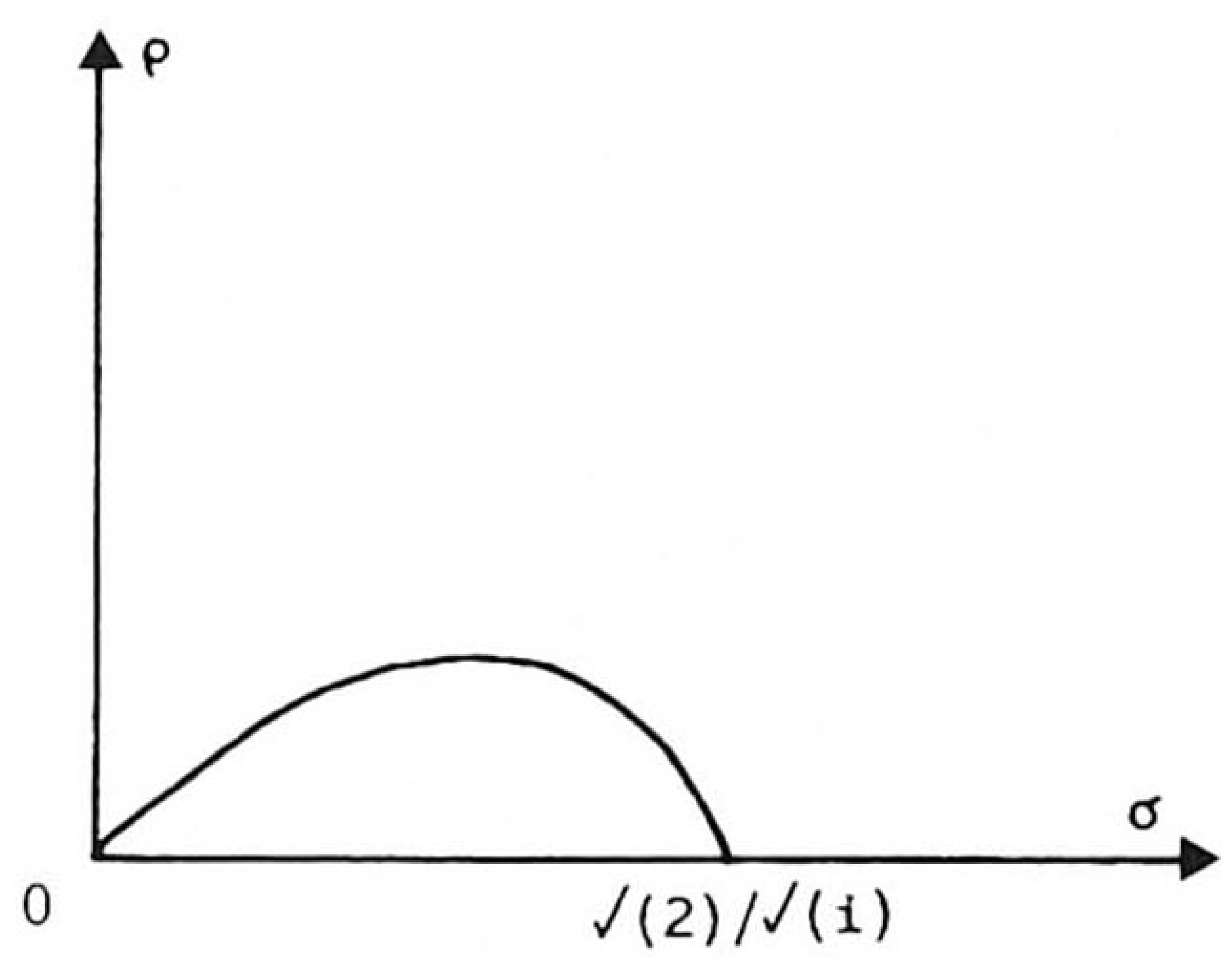

FIGURE 4.2
The relation between ρ and σ.

5. EXTENSION

In this section we will first present a summary of the theory of the static Capital Asset Pricing Model (CAPM), see for instance Copeland & Weston (1983). In 5.2 we apply the results of this model to the stochastic dynamic model of the firm of Bensoussan & Lesourne.

5.1. CAPM model

We assume that the shareholder is a risk-averter and that we deal with a single time period.

We want to value an asset which has a risky payoff at the end of the period, denoted by P_e. The expected return on an investment in the risky asset is determined by the price investors are willing to pay at the beginning of the time period for the right to the risky end-of-period pay off. If P_0 is today's price, the risky return, R_j, equals:

$$R_j = \frac{P_e - P_0}{P_0} \tag{25}$$

The CAPM can be used to determine what the current value of the asset, P_0, should be. The CAPM is :

$$E(R_j) = R_f + [\ E(R_m) - R_f\ ]\ \frac{COV(R_j, R_m)}{VAR(R_m)} \tag{26}$$

in which :

R_f = rate of return of a risk-free asset

R_m = rate of return of the market portfolio, that is a portfolio in which all assets are held according to their value weights.

Defining:

$$\lambda = (\ E(R_m) - R_f)/VAR(R_m) \tag{27}$$

, we get after substituting (27) in (26),(26) in (25) and some rearranging :

$$P_0 = \frac{E(P_e)}{1 + R_f + \lambda COV(R_j, R_m)} \tag{28}$$

which is often referred to as the risk-adjusted rate of return valuation formula. The numerator is the expected end-of-period price for the risky asset and the denominator can be thought of as the discount rate. If the asset has no risk, then its covariance with the market will be zero and the appropriate one-period discount rate is $1 + R_f$, that is 1 + the risk-free rate. For risky assets a risk premium, $\lambda COV(R_j, R_m)$, is added to the risk-free rate so that the discount rate is risk-adjusted.

5.2 The price of risk within a dynamic setting

From the above we learn that risk has its price. Also within the dynamic model it is easy to understand that the shareholder wants an extra return for the risk he takes. Therefore, his time preference rate has to be risk-adjusted. In this way, the next relation arises :

$$i_1 = i + f(\sigma) \tag{29}$$

in which:

i_1 = the risk-adjusted time preference rate of the shareholder

where $f(0) = 0$ and the further shape of the function $f(\sigma)$ depends on the attitude of the shareholder towards risk : the more he is a risk-averter, the faster f will increase. Of course, if he is a "risk-lover", f is a decreasing function of σ. Below, we assume that the shareholder is a risk-averter.

Note that this coincides perfectly with a statement of the previous section, which says that $1/i$ is a measure of the degree to which the shareholder likes risk. However, the solutions of section 4 are not correct. The solutions in Panel A through E deal with decreasing risk, respectively. When we apply equation (29), the time preference rate also falls with the risk and this has the implication that the asymptote of the boundary between the "investment-region" and the "dividend-region", which is a horizontal line on the level $K = K^*$ with K^* such that $\frac{dS}{dK}\big|_{K=K^*} = i$, will be situated on a higher level (note that $S(K)$ is a concave function).

6. CONCLUSION

In this paper we considered especially the impact of uncertainty on the optimal dynamic investment policy of a value maximizing firm. For this purpose, we extended the results of a model of Bensoussan & Lesourne in both the economic and mathematical area. It turns out that with the appearance of uncertainty the amount of cash plays an important role in the optimal solution together with the relation between the attitude of the shareholder towards risk and the level of uncertainty of the firm's revenue. At last, the dynamic model is extended by incorporating the risk-adjusted rate of return valuation formula of the Capital Asset Pricing Model.

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APPENDIX 1. DERIVATION OF THE OPTIMAL SOLUTIONS AND THEIR CRITERIA

In this Appendix we treat the following subjects :

A. The optimal solution if $1/i - \sigma/\surd(2i) \leq 0$

B. The optimal solution if $1/i - \sigma/\surd(2i) > 0$:

B1. The boundary between the "cash-region" and the "dividend-region".

B2. The boundary between the "investment-region" and the "dividend-region".

B3. The boundary between the "cash-region" and the "investment-region"

ad A. <u>The optimal solution if $1/i - \sigma/\surd(2i) < 0$</u>

We first state that the "investment-region" cannot have in its interior a part of the K axis, since there $U(0,K) = 0$, which is contradictory with the condition $\frac{\partial U}{\partial K} \geq 1$.

Also, the "investment-region" will not be above the horizontal line $K = K^*$, with K^* such that $\frac{dS}{dK}|_{K=K^*} = i$, because then the marginal return is lower than the shareholder's time preference rate.

For the fact that in this case the "cash-region" does not exist we refer to the proof of Bensoussan & Lesourne (1980,p268).

In the "dividend-region" U is given by (22). Since this region covers the K axis and $U(0,K) = 0$:

$$0 = \frac{S(K)}{i} + c_1(K) + c_2(K) \tag{30}$$

From (12) we derive that U always has a finite value and therefore $c_1(K) = 0$. Due to (30) we can conclude that $c_2(K) = -\frac{S(K)}{i}$, which implies :

$$U = \frac{S(K)}{i} - \frac{S(K)}{i} \exp\left(- \frac{M\sqrt{(2i)}}{\sigma S(K)} \right) \tag{31}$$

On the boundary between the "dividend-region" and the "investment-region" it must hold that $\frac{\partial U}{\partial K} = 1$.

$$\frac{\partial U}{\partial K} = \left(\frac{dS}{dK}/i\right) \left[1 - \left(1 + \frac{M\sqrt{(2i)}}{\sigma S(K)}\right) \exp\left(- \frac{M\sqrt{(2i)}}{\sigma S(K)} \right)\right] \tag{32}$$

Due to (32) we get :

$$\text{for } M \to \infty : \frac{\partial U}{\partial K} = 1 \text{ when } \frac{dS}{dK} = i \tag{33}$$

Now, we prove that the boundary is an increasing function in the plane(M,K). From (32) we derive :

$$\frac{\partial^2 U}{\partial K \partial M} = 2 \ \frac{dS}{dK} M/(\sigma^2 (S(K))^2) \exp\left(- \frac{M\sqrt{(2i)}}{\sigma S(K)} \right) > 0 \tag{34}$$

$$\frac{\partial^2 U}{\partial K^2} = \frac{d^2 S}{dK^2} \frac{\partial U}{\partial K}/\left(\frac{dS}{dK}\right) - \frac{2M^2 \left(\frac{dS}{dK}\right)^2}{\sigma^2 (S(K))^3} \exp\left(- \frac{M\sqrt{(2i)}}{\sigma S(K)} \right) < 0 \tag{35}$$

From (34) we derive that $\frac{\partial U}{\partial K}$ increases if M increases. On the whole boundary $\frac{\partial U}{\partial K}$ must be equal to 1, so when M increases we have to find a K which liquidates the increase in $\frac{\partial U}{\partial K}$ due to the increase in M. From (35) we can conclude that $\frac{\partial U}{\partial K}$ decreases if K increases, so at the boundary a higher level of M corresponds to a higher level of K and therefore the boundary is an increasing function in the plane (M,K).

ad B The optimal solution if $1/i - \sigma/\sqrt{(2i)} > 0$

B1 The boundary between the "cash-region" and the "dividend-region"

First, we prove the existence of the "cash-region" when $1/i - \sigma/\sqrt{(2i)} > 0$ and $\sigma > 0$:
The "investment-region" does not cover the K axis, because of the contradiction between $U(0,K) = 0$ for every K and the condition $\frac{\partial U}{\partial K} \geq 1$.
If the "dividend-region" covers the K axis, U is given by (31) and from this equation we derive :

$$\frac{\partial U}{\partial M} = \sqrt{(2)}/(\sigma\sqrt{(i)}) \exp\left(- \frac{M\sqrt{(2i)}}{\sigma S(K)} \right) \tag{36}$$

Due to (36), we can conclude that $\frac{\partial U}{\partial M} > 1$ for $M = 0$, so the "dividend-region" cannot have in its interior a part of the K axis. Therefore the only region which can cover the K axis is the "cash-region".

From the above and (17) we get :

$$k_1(K) + k_2(K) = 0 \rightarrow k_2(K) = - k_1(K) \tag{37}$$

From the facts that U always has a finite value and the firm carries out a dividend policy if M is great and $K > K^*$, we can derive that :

$$c_1(K) = 0 \tag{38}$$

We assume that $M = \gamma(K)$ is the boundary between the "cash-region" and the "dividend-region", so :

$M < \gamma(K) \Rightarrow$ cash policy and $M > \gamma(K) \Rightarrow$ dividend policy

At the boundary two conditions must hold :

- equality of (17) and (22)
- $\frac{\partial U}{\partial M} = 1$

According to these conditions and (37) and (38), we get :

$$k_1(K)[\exp(\frac{r_1\gamma(K)}{S(K)}) - \exp(\frac{r_2\gamma(K)}{S(K)})] = \frac{S(K)}{i} + c_2(K)\exp(-\frac{\gamma(K)\sqrt{(2i)}}{\sigma S(K)}) \tag{39}$$

$$k_1(K)[\frac{r_1}{S(K)}\exp(\frac{r_1\gamma(K)}{S(K)}) - \frac{r_2}{S(K)}\exp(\frac{r_2\gamma(K)}{S(K)})]=1=-\frac{c_2(K)\sqrt{(2i)}}{\sigma S(K)}\exp(-\frac{\gamma(K)\sqrt{(2i)}}{\sigma S(K)}) \tag{40}$$

After assuming $\gamma(K) = \rho(K)S(K)$ and some rearranging we can rewrite (39) and (40) into :

$$k_1(K)[\exp(r_1\rho(K)) - \exp(r_2\rho(K))] = c_2(K)\ \exp(-\frac{\rho(K)\sqrt{(2i)}}{\sigma})(1 - \frac{\sqrt{(2i)}}{\sigma i}) \tag{41}$$

$$k_1(K)[r_1\exp(r_1\rho(K)) - r_2\exp(r_2\rho(K))] = c_2(K)\ \exp(-\frac{\rho(K)\sqrt{(2i)}}{\sigma})(-\frac{\sqrt{(2i)}}{\sigma}) \tag{42}$$

From (41) and (42) it is easily deduced that :

$$\exp((r_1-r_2)\rho(K)) = [1 - r_2(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}})]/[1 - r_1(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}})] \tag{43}$$

Due to (43) we can conclude that ρ is independent of K, so the boundary between the cash- and the dividend-region is equal to :

$$M = \rho S(K) \tag{44}$$

From (40) we can conclude that the functions $k_1(K)$ and $c_2(K)$ are given by the following expressions :

$$k_1(K) = S(K)/(r_1\ \exp(r_1\rho) - r_2\ \exp(r_2\rho)) \tag{45}$$

$$c_2(K) = -\frac{\sigma S(K)}{\sqrt{(2i)}}\ \exp(\frac{\rho\sqrt{(2i)}}{\sigma}) \tag{46}$$

If equation (44) is the real boundary, the following conditions have to hold :

- if $M \geq \rho S(K)$: 1. $\frac{\partial U}{\partial M} \leq 1$
- if $M \leq \rho S(K)$: 2. $\frac{\partial U}{\partial M} \geq 1$

  3. $\frac{\partial U}{\partial M} \geq \frac{\partial U}{\partial K}$

Por the proof of the satisfaction of the first two conditions we refer to

Bensoussan & Lesourne (1980, p266). We only prove the third condition. By easy computations this amounts to checking that (due to (17), (37) and (45)):

$$[r_1 - \frac{dS}{dK}(1 - \frac{r_1 M}{S(K)})]\exp(\frac{r_1 M}{S(K)}) > [r_2 - \frac{dS}{dK}(1 - \frac{r_2 M}{S(K)})]\exp(\frac{r_2 M}{S(K)}) \tag{47}$$

Setting Z = M/S(K) and :

$$\psi(Z) = [r_1 - \frac{dS}{dK}(1 - r_1 Z)]\exp(r_1 Z) - [r_2 - \frac{dS}{dK}(1 - r_2 Z)]\exp(r_2 Z) \tag{48}$$

$$\frac{d\psi}{dZ} = (1 + Z\frac{dS}{dK})[r_1^2 \exp(r_1 Z) - r_2^2 \exp(r_2 Z)] \tag{49}$$

Obviously, $\frac{d\psi}{dZ} \leq 0$ if $Z < \rho$, whatever the value of K. Now it suffices to check that $\psi(\rho) \geq 0$:

$$\psi(\rho) = [(1 + \rho\frac{dS}{dK})/(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}}) - \frac{dS}{dK}][\exp(r_1\rho) - \exp(r_2\rho)] \tag{50}$$

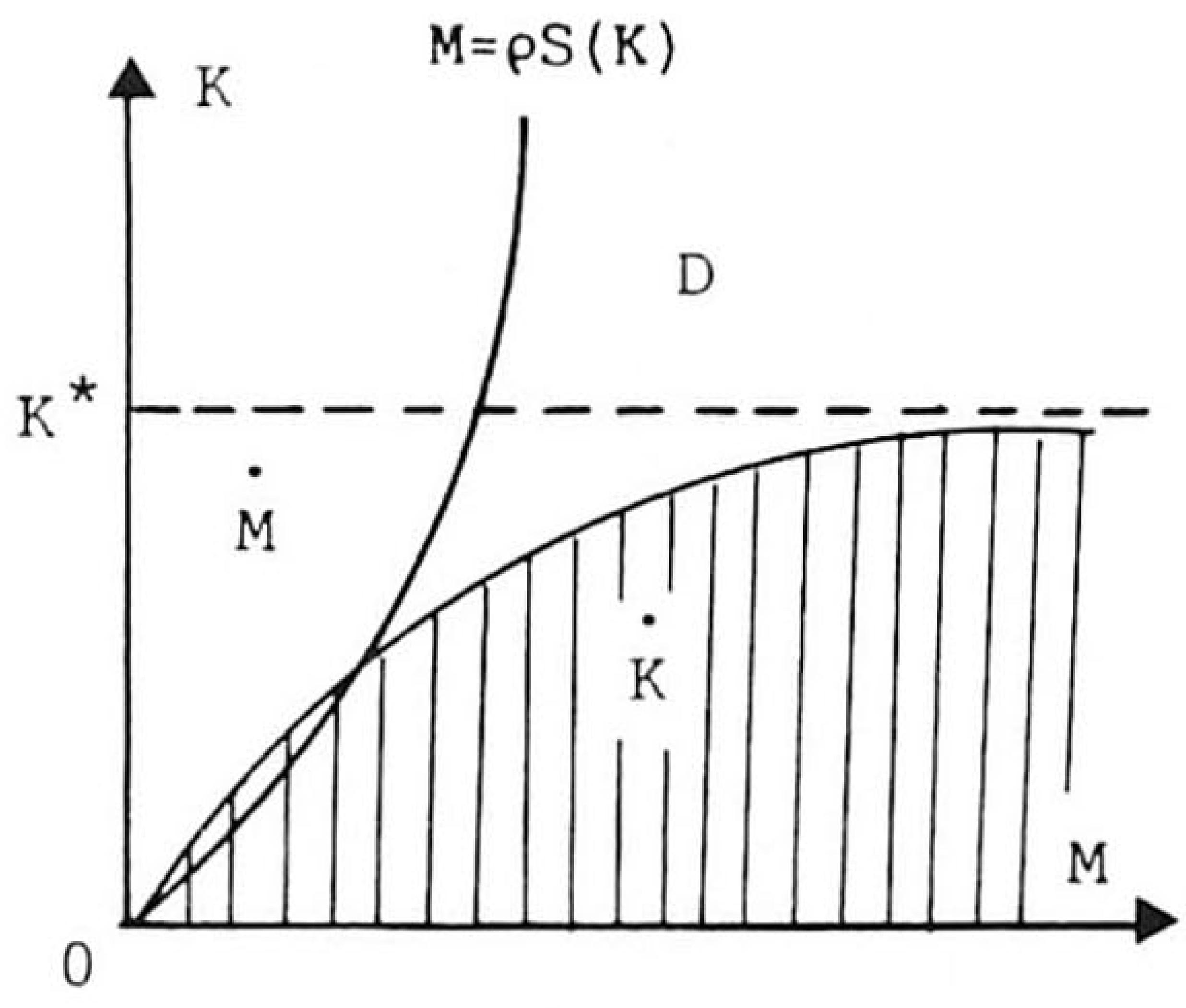
If $\rho/(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}}) \geq 1$, it is obvious that $\psi(\rho) \geq 0$. If $\rho < \frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}}$, then:

$$\psi(\rho) \geq 1/(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}}) - \frac{dS}{dK}(K=0)(1 - \rho/(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}})) \geq 0 \tag{51}$$

We can rewrite (51) into :

$$\psi(\rho) \geq 1/(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}}) - \frac{dS}{dK}(K=0)(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}} - \rho)/(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}}) \geq 0 \tag{52}$$

(52) holds if : $\frac{dS}{dK}(K=0)(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}} - \rho) \leq 1$ (see the "Panel B and C solutions" in section 4). In solution 4 of section 4 the condition $\frac{\partial U}{\partial M} \geq \frac{\partial U}{\partial K}$ does not need to hold in the whole area in which $M \leq \rho S(K)$, because it is possible that in a part of this area the firm carries out an investment policy (see figure A.1).



$$1/i - \sigma/\sqrt{(2i)} - \rho > \frac{1}{\left.\frac{dS}{dK}\right|_{K=0}}$$

FIGURE A.1

The "Panel D solution" of section 4.

B2 <u>The boundary between the "investment-region" and the "dividend-region"</u>

After substituting (38) and (46) in (22), we get that in the "dividend-region" it holds that :

$$U = S(K)/i - \frac{\sigma S(K)}{\sqrt{(2i)}} \exp\left(\left(\rho - \frac{M}{S(K)}\right)\frac{\sqrt{(2i)}}{\sigma}\right) \tag{53}$$

From (53) we derive :

$$\frac{\partial U}{\partial K} = \frac{dS}{dK}\left[\frac{1}{i} - \left(\frac{\sigma}{\sqrt{(2i)}} + \frac{M}{S(K)}\right)\exp\left(\left(\rho - \frac{M}{S(K)}\right)\frac{\sqrt{(2i)}}{\sigma}\right)\right] \tag{54}$$

$$\frac{\partial^2 U}{\partial K^2} = \frac{d^2 S}{dK^2}\frac{\partial U}{\partial K}/\left(\frac{dS}{dK}\right) - \left(\frac{dS}{dK}\right)^2 \frac{M^3}{(S(K))^3}\exp\left(\left(\rho - \frac{M}{S(K)}\right)\frac{\sqrt{(2i)}}{\sigma}\right) < 0 \tag{55}$$

$$\frac{\partial^2 U}{\partial K \partial M} = \frac{dS}{dK}\frac{M}{(S(K))^2}\frac{\sqrt{(2i)}}{\sigma} \exp\left(\left(\rho - \frac{M}{S(K)}\right)\frac{\sqrt{(2i)}}{\sigma}\right) > 0 \tag{56}$$

At the boundary it must hold that : $\frac{\partial U}{\partial K} = 1$ (57)

After following the same reasoning as in ad A (the case $1/i - \sigma/\sqrt{(2i)} \leq 0$), we can conclude that this boundary also increases in the plane (M,K). Due to (54) we can show that the boundary increases between a K such, that $\frac{dS}{dK}\left(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}} - \rho\right) = 1$ and a K such, that $\frac{dS}{dK}\frac{1}{i} = 1$ when M varies between

$\rho S(K)$ and ∞. So, at the intersection point of the investment-dividend boundary and the cash-dividend boundary the following must hold:

$$M = \rho S(K) \text{ and } \frac{dS}{dK}\left(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}} - \rho\right) = 1 \tag{58}$$

From (58) and the concavity of S(K) we can easily derive the following conclusions :

- The boundaries do not intersect if $\frac{dS}{dK}(K=0)\left(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}} - \rho\right) < 1$ (Panel B)

- The boundaries intersect in the origin if $\frac{dS}{dK}(K=0)\left(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}} - \rho\right) = 1$ (Panel C)

- The boundaries intersect in a point at which $K > 0$ if $\frac{dS}{dK}(K=0)\left(\frac{1}{i} - \frac{\sigma}{\sqrt{(2i)}} - \rho\right) > 1$ (Panel D)

B3 The boundary between the "cash-region" and the "investment-region"

We prove that this boundary intersects the origin. To do so, we show that the "investment-region" cannot have in its interior a part of the K axis and the "cash-region" does not cover the M axis :

- The "investment-region" does not in its interior have a part of the K axis because of the contradiction between U(0,K) for every K and the condition $\frac{\partial U}{\partial K} \geq 1$.

- At the M axis it holds that $K = 0$, so $M > \rho S(K)$. But then a dividend-policy is better than a cash policy, so the "cash-region" does not cover the M axis.

APPENDIX 2. THE RELATION BETWEEN σ AND ρ

After substituting (18) and (19) in (23) we obtain the following equation:

$$\exp\left(\frac{2\rho\sqrt{(1 + 2\sigma^2 i)}}{\sigma^2}\right) = \frac{[\sigma^2 + (1 + (1 + 2\sigma^2 i)^{1/2})(1/i - \sigma/\sqrt{(2i)})]}{[\sigma^2 + (1 - (1 + 2\sigma^2 i)^{1/2})(1/i - \sigma/\sqrt{(2i)})]} \tag{59}$$

When we differentiate (59) to σ, we get after substituting (59) into this derivative and after many computations :

$$\frac{d\rho}{d\sigma}(4i\sigma^3 + 2\sigma) = (4i\rho - 3)\sigma^2 + 4\rho - \sigma^3\surd(2i) \tag{60}$$

After differentiating (60) again to σ, we get an expression for $\frac{d^2\rho}{d\sigma^2}$:

$$\frac{d^2\rho}{d\sigma^2}(4i\sigma^3 + 2\sigma)^2 = -4i\sigma^5\surd(2i) - 8\sigma^3\surd(2i) - (8i\rho + 18)\sigma^2 + 8\rho \tag{61}$$

From (60) and (61) we derive for $\sigma > 0$:

$$\frac{d\rho}{d\sigma} \gtreqless 0 \text{ if } (4i\rho - 3)\sigma^2 + 4\rho - \sigma^3\surd(2i) \gtreqless 0 \tag{62}$$

$$\frac{d^2\rho}{d\sigma^2} \gtreqless 0 \text{ if } -4i\sigma^5\surd(2i) - 8\sigma^3\surd(2i) - (8i\rho + 18)\sigma^2 + 8\rho \gtreqless 0 \tag{63}$$

We know that there is only a "cash-region" for $\sigma = 0$ between 0 and $\surd(2)/\surd(i)$, so ρ is only positive in this σ-region and $\rho = 0$ for $\sigma = 0$ and $\sigma = \surd(2)/\surd(i)$. Due to this reasoning and (62) and (63) we are able to construct the following figure :

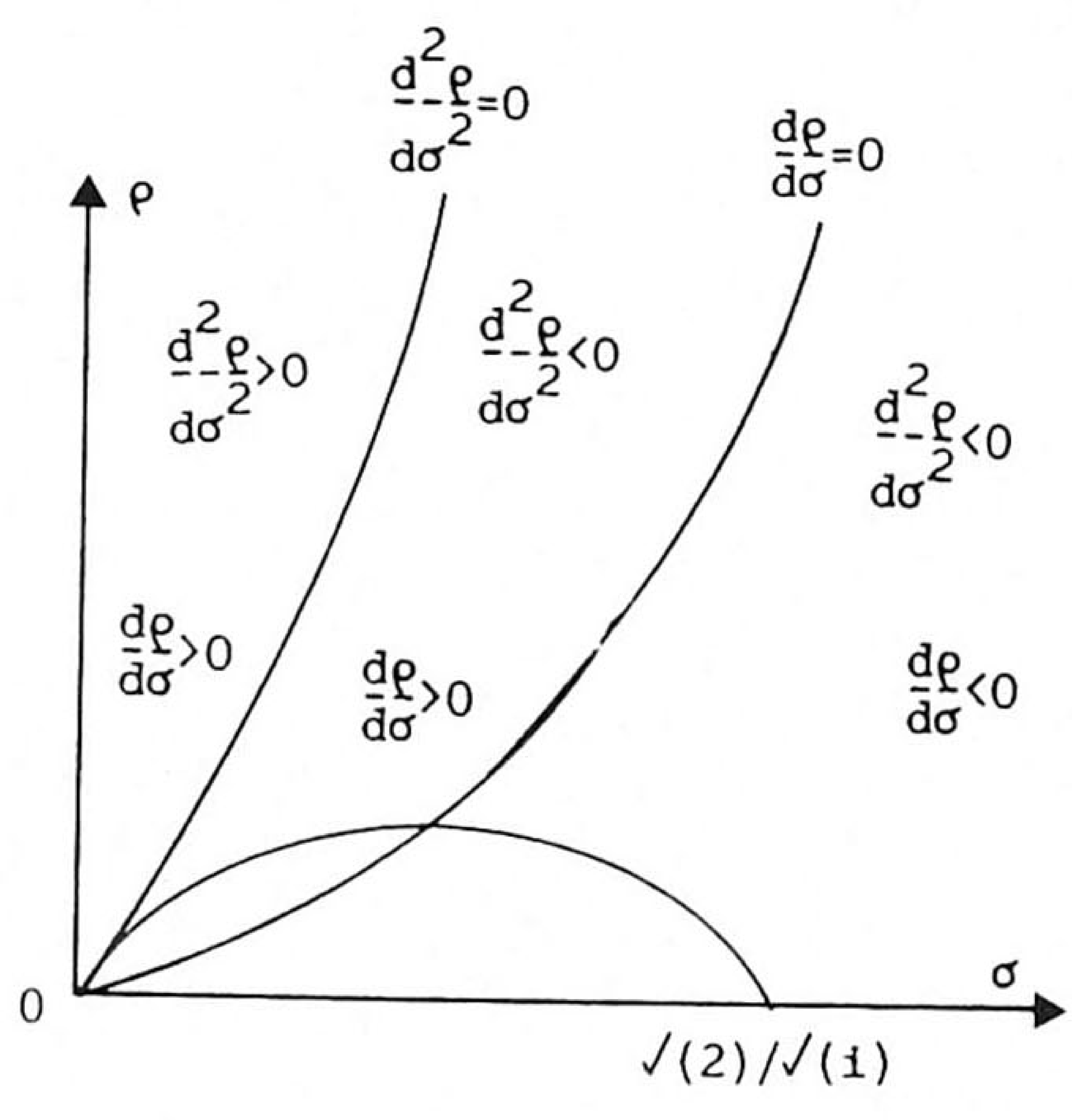


FIGURE A.2

The relation between ρ and σ.